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Full counting statistics of a general quantum mechanical variable

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Abstract. We present a quantum mechanical framework for defining the statistics of measurements of $\int dt\, \hat{A}(t)$, $A(t)$ being a quantum mechanical variable. This is a generalization of the so-called full counting statistics proposed earlier for DC electric currents. We develop an influence functional formalism that allows us to study the quantum system along with the measuring device while fully accounting for the back action of the detector on the system to be measured. We define the full counting statistics of an arbitrary variable by means of an evolution operator that relates the initial and final density matrices of the measuring device. In this way we are able to resolve inconsistencies that occur in earlier definitions. We suggest two schemes to observe the so defined statistics experimentally.

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1 Introduction

The measurement paradigm in quantum mechanics assumes that a measurement is done instantly [1]. This is in contrast with a realistic measurement of, say, an electric current, where the result of measurement is averaged over a sufficiently long time interval. If one intends to measure a variable A, the individual measurement gives $\int_0^\tau A(t)dt/\tau$. The reason for this is obvious: any measurement has to be accurate. The integration over time averages over instant fluctuations of $A(t)$ resulting in a more accurate outcome of an individual measurement of this sort. The dispersion of the probability distribution of the outcomes is supposed to vanish in the limit of $\tau \to \infty$. This paper focuses on the problems related to the determination of this probability distribution, the statistics of the measurement results.

Several years ago Levitov and Lesovik [2–4] made a significant step in the understanding of this fundamental issue. They introduced the concept of full counting statistics (FCS) of electric current and have found this statistics for the generic case of a one-mode mesoscopic conductor. The word "counting" reflects the discreteness of the electric charge. If electrons were classical particles, one could just count electrons traversing a conductor. The FCS could be readily defined in terms of the probability to have N electrons transferred through the conductor during a time interval τ, $P_\tau(N)$. With this distribution function one calculates the average current $\langle N\rangle/\tau$, current noise $(\langle N\rangle^2 > -\langle N\rangle^2)/\tau$ and all higher cumulants of the current. A non-trivial value of interest is the probability to have big deviations from the average value. This can be measured with a threshold detector. The probability distribution $P_\tau(N)$ would be the goal of a quantum-mechanical calculation.

The operator of electric current through a conductor, $\hat{I}$, is well-defined in the Fock space spanned by the scattering states of electrons. The initial idea of Lesovik and Levitov [2] was to define an operator of transferred charge by means of a seemingly obvious relation

$$\hat{Q}_{tr} = \int_0^\tau dt\, \hat{I}(t). \tag{1}$$

To this operator one applies the general paradigm of quantum measurement [1]: The probability to have a certain charge q transferred equals the square of the projection of the wave function of the system on the eigenstate of $\hat{Q}_{tr}$ with eigenvalue q. Lesovik and Levitov were able to perform the challenging calculation of these projections. However, they were hardly satisfied with the results. For instance, the transferred charge was not quantized in units of the elementary charge.

This is why in their subsequent paper [3] the same authors proposed another method of evaluating $P_\tau(N)$. Their scheme invoked a measuring device. As a model device, they chose a precessing spin-1/2 whose precession angle should be proportional to the transferred charge.

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The measurement paradigm is then applied to the device. In this way they were able to obtain a satisfactory definition of the statistics $P_\tau(N)$ with an integer number of charges transferred. The details of the calculation and a thorough discussion are presented in [4].

It was clear to the authors of [4] that their definition of the FCS does not depend on a specific measurement scheme. However, this fact was not explicitly evident. For several years this hindered the impact of these outstanding contributions.

One of the authors has recently proposed a slightly different calculation scheme of FCS that does not invoke any measuring device but still leads to the same results [5]. The observation was that the cumulants of the current can be obtained as non-linear responses of a system to a fictitious field that can only be defined in the framework of the Keldysh diagrammatic technique [6]. The calculation of FCS can be accomplished with a slight extension of the Keldysh technique. This meant some progress since the methods of the Keldysh technique are well elaborated and can be readily applied to a variety of physical systems and situations.

More recently, it has been shown, that a statistics very similar to the one defined in [4] can also be obtained without explicitly modeling a charge detector, as a property of the current conductor only [7].

In [8] the charge transfer between two superconductors has been addressed. The problem can be tackled with an extension of the above-mentioned Keldysh technique. The expressions for $P_\tau(N)$ were obtained. Albeit the authors have encountered a significant difficulty in understanding the results in classical terms, using the schemes proposed in [3,7]. The calculation gave *negative* probabilities. This indicates that the results cannot be interpreted without invoking a quantum description of a detector.

All this suggests that the quantum mechanical concept of counting statistics shall be refined and the generality of previously used definitions shall be accessed. This is done in the present article.

To preserve generality, we analyze the counting statistics of an arbitrary quantum mechanical variable A. Then the result does not have to be discrete, and, strictly speaking, no counting takes place. We keep the term "counting" for historical reasons.

We introduce a detector whose read-off we can interpret as the statistics of $\int dt\, \hat{A}(t)$ and we determine its quantum mechanical time evolution. It turns out, that the answer does not depend on the details of the detector. This allows for a formal separation of the measured system from the measuring device. We develop an exact quantum mechanical description of the measurement setup in terms of a path integral over detector variables and derive our results from this description. We show that a classical interpretation of FCS is only possible in the presence of a certain symmetry. For the FCS of the electric current, this symmetry is gauge invariance. The probability distribution reduces in this case to the form found in [4]. In superconductors gauge invariance is broken and the FCS must be interpreted along quantum mechanical lines.

It is the main message of our paper that this interpretation problem does not make the concept of full counting statistics useless and/or unphysical. We show that it is the FCS that completely determines the evolution of the density matrix of the detector. We show thereby that the statistics is observable in experiments. We propose and discuss two concrete measuring schemes.

The paper is organized as follows. We start with a general compact discussion of the interpretation problems. We present our detection model in Section 3. It is analyzed in the subsequent section. Section 5 defines the FCS and gives its interpretation. The subsequent sections provide examples of the FCS for a system in the ground state, a normal conductor and a harmonic oscillator. We characterize the FCS in Sections 9 and 10 where two concrete schemes are discussed that allow it to be measured experimentally.

2 General discussion

It is not a priori clear why the operator definition (1) produces senseless results. We list below possible intuitive reasons for this. To start with, the paradigm concerns an *instant* measurement. The operator definition (1) is not local in time and accumulates information about the quantum state of the system for a (long) interval of time. The applicability of the paradigm is therefore not obvious. For instance, the averages of powers of $\hat{Q}_{tr}$ can be expressed in terms of correlators of currents

$$\langle \hat{Q}_{tr}^N \rangle = \int_0^\tau dt_1 ... dt_N \langle \hat{I}(t_1)\hat{I}(t_2)...\hat{I}(t_N)\rangle \cdot \qquad (2)$$

Usually causality comes into quantum mechanics via time ordering of operator products. There is no time ordering of current operators in (2). This may indicate implicit problems with causality. The second reason is as follows. It seems obvious that the time integral of $\hat{I}$ can be associated with a physical operator of charge. For an arbitrary operator $\hat{A}$ it may be difficult to find such a physical associate. Still, integrals of $\hat{A}$ can be measured, and the statistics of them can be accumulated.

In view of this problem it seems to be necessary to model the measurement process in order to define a statistics of time averages. This has been done in [3] by introducing the spin-1/2 detector. Since within this detection model the described interpretation problems ("negative probabilities") [8] arose, here we will refine the model [9]. We adopt a detector model that has already been used by John von Neumann in an analysis of the quantum measurement process [10]. We introduce a detector variable x, whose operator $\hat{x}$ commutes with all operators of the system to be measured. We assume that the canonically conjugated variable, q, ($[\hat{x}, \hat{q}] = i$, in units with $\hbar = 1$) can be measured according to the paradigm. Next we introduce an interaction between the system and the detector in a way that in the time interval $(0, \tau)$ the Heisenberg equation of motion reads

$$\dot{\hat{q}}(t) = \hat{A}(t), \qquad (3)$$

simulating equation (1). In this way we avoid all possible difficulties of misinterpreting the paradigm. The integral of $A(t)$ is now correctly associated with an operator that can *a priori* be measured.

However, there is a price to pay. As we show below, the FCS can be defined in this way as an operator that relates the density matrices of the detector before and after the measurement. In general, it is not the same as the probability distribution of shifts of the detector momentum which can be associated with probabilities of $\int_0^\tau dt\,\hat{A}(t)$ in the classical limit. The FCS can be interpreted in such terms only under certain conditions, which are satisfied for the statistics of current in normal metal conductors.

To make the detector more realistic and thus show the generality of the results, one shall introduce some internal dynamics of the detector variable. These dynamics would make the detector a non-ideal one: the readings may differ from the definitions (1) and (3). The path integral approach we describe below provides the most convenient way to incorporate these internal dynamics.

3 Model

The detector in our model consists of one degree of freedom x (with a conjugated variable q) with the Hamiltonian $\hat{q}^2/2m$. The system shall be coupled to the position x of the detector during the time interval $[0,\tau]$ and be decoupled adiabatically for earlier and later times. For this we introduce a smooth coupling function $\alpha_\tau(t)$ that takes the value 1 in the time interval $[0,\tau]$ and is zero beyond the interval $[t_1,t_2]$ ($t_1 < 0$ and $t_2 > \tau$). The values for $t_1 < t < 0$ and $\tau < t < t_2$ are chosen in a way that provides an adiabatic switching. The entire Hamiltonian reads then

$$H(t) = \hat{H}_{\rm sys} - \alpha_\tau(t)\hat{x}\hat{A} + \frac{\hat{q}^2}{2m}\cdot \tag{4}$$

The Heisenberg equation of motion for the detector momentum q

$$\dot{\hat{q}}(t) = \alpha_\tau(t)\hat{A}(t) \tag{5}$$

suggests, that the statistics of outcomes of measurements of the detector's momentum after having it uncoupled from the system corresponds to the statistics of the time average $\int_0^\tau dt\,\hat{A}(t)$ that we are interested in.

The coupling term can be viewed as a disturbance of the system measured by the detector. To minimize this disturbance, one would clearly like to concentrate the detector wave function around $x = 0$. The uncertainty principle forbids, however, to localize it completely. Thereby one would loose all information about the detector momentum, which is to be measured. This is a fundamental limitation imposed by quantum mechanics, and we are going to explore its consequences step by step. To discern it from a classical back action of the detector we take the limit of a static detector with $m \to \infty$, such that $\dot{\hat{x}} = 0$ and any classical back action is ruled out.

4 Approach

To predict the statistics of measurement outcomes in our detection model we need the reduced density matrix of the detector after the measurement, at $t > t_2$. If there were no system to measure we could readily express it in the form of a path integral in the (double) variable $x(t)$ over the exponential of the detector action. This is still possible in the presence of a system coupled to the detector [11]. The information about the system to be measured can be compressed into an extra factor in this path integral, the so-called influence functional. This makes the separation between the detector and the measured system explicit. To make contact with [4], we write the influence functional as an operator expression that involves system degrees of freedom only. We denote the initial detector density matrix (at $t < t_1$) by $\rho^{in}(x^+,x^-)$ and the final one (at $t > t_2$, after having traced out the system's degrees of freedom) by $\rho^f(x^+,x^-)$. $\hat{R}$ denotes the initial density matrix of the system. The entire initial density matrix is assumed to factorize, $\hat{D} = \hat{R}\hat{\rho}^{in}$.

We start out by inserting complete sets of states into the expression for the time development of the density matrix

$$\rho^f(x^+,x^-) = \mathop{\rm Tr}_{\rm System} \left\langle x^+ \left| \overrightarrow{T} e^{-i\int_{t_1}^{t_2} dt\,\left[\hat{H}_{\rm sys} - \alpha_\tau(t)\hat{x}\hat{A} + \hat{q}^2/2m\right]} \right.\right. \\ \left.\left. \times \hat{D}\overleftarrow{T} e^{i\int_{t_1}^{t_2} dt\,\left[\hat{H}_{\rm sys} - \alpha_\tau(t)\hat{x}\hat{A} + \hat{q}^2/2m\right]} \right| x^- \right\rangle\cdot \tag{6}$$

Here, $\overrightarrow{T}$ ($\overleftarrow{T}$) denotes (inverse) time ordering. As complete sets of states, we choose product states of any complete set of states of the system and alternatingly complete sets of eigenstates of the position or the momentum operator of the detector. These intermediate states allow us to replace the position and momentum operators in the time development exponentials by their eigenvalues. We can then do the integrals over the system states as well as the momentum integrals and arrive at the expression

$$\rho^f\left(x^+,x^-\right) = \\ \int_{x^+(t_2)=x^+\ x^-(t_2)=x^-} \mathcal{D}\left[x^+\right]\ \mathcal{D}\left[x^-\right]\ \rho^{in}\left[x^+(t_1),x^-(t_1)\right] e^{-i\mathcal{S}_{\rm Det}\left(\left[x^+\right],\left[x^-\right]\right)} \\ \times \mathop{\rm Tr}_{\rm System} \overrightarrow{T} e^{-i\int_{t_1}^{t_2} dt\left[\hat{H}_{\rm sys} - \alpha_\tau(t)x^+(t)\hat{A}\right]} \\ \times R\,\overleftarrow{T} e^{i\int_{t_1}^{t_2} dt\left[\hat{H}_{\rm sys} - \alpha_\tau(t)x^-(t)\hat{A}\right]} \tag{7}$$

with the detector action

$$\mathcal{S}_{\rm Det}([x^+],[x^-]) = -\int_{t_1}^{t_2} dt\,\frac{m}{2}\left[(\dot{x}^+)^2 - (\dot{x}^-)^2\right]. \tag{8}$$

We rewrite the expression as

$$\rho^f(x^+,x^-) = \int dx_1^+ dx_1^-\ K(x^+,x^-;x_1^+,x_1^-,\tau)\rho^{in}(x_1^+,x_1^-) \tag{9}$$

with the kernel

$$K(x^+,x^-;x_1^+,x_1^-,\tau)=\int_{x^+(t_2)=x^+,x^+(t_1)=x_1^+}\mathcal{D}[x^+]\int_{x^-(t_2)=x^-,x^-(t_1)=x_1^-}\mathcal{D}[x^-]\times\mathcal{Z}_{\rm Sys}([\alpha_\tau x^+],[\alpha_\tau x^-])\,e^{-i\mathcal{S}_{\rm Det}([x^+],[x^-])} \quad (10)$$

that contains the influence functional

$$\mathcal{Z}_{\rm Sys}([\chi^+],[\chi^-])=\underset{\rm System}{\rm Tr}\ \overrightarrow{T}e^{-i\int_{t_1}^{t_2}dt\left[\hat{H}_{\rm sys}-\chi^+(t)\hat{A}\right]}\hat{R}\ \overleftarrow{T}\ e^{i\int_{t_1}^{t_2}dt\left[\hat{H}_{\rm sys}-\chi^-(t)\hat{A}\right]}. \quad (11)$$

Taking the limit of an infinite detector mass, we find that $\mathcal{S}_{\rm Det}$ in equation (10) suppresses all fluctuations in the path integral. In the Appendix we show that the kernel $K(x^+,x^-,x_1^+,x_1^-,\tau)$ becomes local in position space,

$$K(x^+,x^-,x_1^+,x_1^-,\tau)=\delta(x^+-x_1^+)\,\delta(x^--x_1^-)\ P(x^+,x^-,\tau) \quad (12)$$

with

$$P(x^+,x^-,\tau)=\underset{\rm System}{\rm Tr}\ \overrightarrow{T}e^{-i\int_{t_1}^{t_2}dt\left[\hat{H}_{\rm sys}-\alpha_\tau(t)x^+\hat{A}\right]}\hat{R}\ \overleftarrow{T}e^{i\int_{t_1}^{t_2}dt\left[\hat{H}_{\rm sys}-\alpha_\tau(t)x^-\hat{A}\right]}. \quad (13)$$

It is constructive to rewrite now the density matrices in the Wigner representation

$$\rho(x,q)=\int\frac{dz}{2\pi}\ e^{-iqz}\ \rho\left(x+\frac{z}{2},x-\frac{z}{2}\right) \quad (14)$$

and define

$$P(x,q,\tau)=\int\frac{dz}{2\pi}e^{-iqz}\ P\left(x+\frac{z}{2},x-\frac{z}{2},\tau\right). \quad (15)$$

This gives the convenient relation

$$\rho^f(x,q)=\int dq_1\ P(x,q-q_1,\tau)\ \rho^{in}(x,q_1). \quad (16)$$

5 Interpretation of the FCS

We adopt the relations (13, 15) and (16) as the definition of the FCS of the variable A. Let us see why. First let us suppose that we can treat the detector classically. Then the density matrix of the detector in the Wigner representation can be interpreted as a classical probability distribution $\Pi(x,q)$ to be at a certain position x with momentum q. This allows for a classical interpretation of $P(x,q,\tau)$ as the probability to have measured $q=\int_0^\tau A(t)$. Indeed, one sees from (15) that the final $\Pi(x,q)$ is obtained from the initial one by shifts in q, $P(x,q,\tau)$ being the probability distribution of those shifts.

In general, the density matrix in the Wigner representation cannot be interpreted as a probability to have a certain position and momentum since it is not positive. Concrete calculations given below illustrate that $P(x,q,\tau)$ does not have to be positive either. Consequently, it cannot be interpreted as a probability distribution. Still it predicts the results of measurements according to equation (16).

There is, however, an important case when a classical interpretation of $P(x,q,T)$ as a probability distribution is indeed applicable. It is the case that $P(x,q,\tau)$ does not depend on x, $P(x,q,\tau)\equiv P(q,\tau)$. Then, integrating equation (16) over x, we find

$$\Pi^f(q)=\int dq'\ P(q-q',\tau)\ \Pi^{in}(q') \quad (17)$$

with $\Pi(q)\equiv\int dx\,\rho(x,q)$. Therefore, the FCS in this special case is the kernel that relates the probability distributions of the detector momentum before and after the measurement, $\Pi^{in}(q)$ and $\Pi^f(q)$, to each other. Those distributions are positive and so is $P(q,\tau)$.

When studying the FCS of a stationary system and the measurement time τ exceeds time scales associated with the system, the operator expression in equation (13) can be seen as a product of terms corresponding to time intervals. Therefore in this limit of $\tau\to\infty$ the dependence on the measuring time can be reconciled into

$$P(x^+,x^-,\tau)=e^{-\mathcal{E}(x^+,x^-)\tau} \quad (18)$$

where the expression in the exponent is supposed to be large. Then the integral (15) that defines the FCS can be evaluated by the saddle point approximation. Defining the time average $\bar{A}=q/\tau$, that is, $\bar{A}$ is the result of a measurement of $\int_0^\tau A(t)dt/\tau$, the FCS can be recast into the from

$$P(x,\bar{A},\tau)=e^{-\tilde{\mathcal{E}}(x,\bar{A})\tau} \quad (19)$$

where $\tilde{\mathcal{E}}$ is defined as the (complex) extremum with respect to (complex) z:

$$\tilde{\mathcal{E}}=\underset{z}{\rm extr}\left\{\mathcal{E}\left(x+\frac{z}{2},x-\frac{z}{2}\right)+i\bar{A}z\right\}\cdot \quad (20)$$

The average value of $\bar{A}$ and its variance (noise) can be expressed in terms of derivatives of $\mathcal{E}$:

$$\langle\bar{A}\rangle=-\lim_{z\to0}\frac{\partial\mathcal{E}\left(x+z/2,x-z/2\right)}{i\partial z};$$
$$\tau\left\langle\left\langle\bar{A}^2\right\rangle\right\rangle=\lim_{z\to0}\frac{\partial^2\mathcal{E}\left(x+z/2,x-z/2\right)}{\partial z^2}\cdot \quad (21)$$

More generally, the quantity $P(x^+,x^-,\tau)$ is the generating function of moments of q. It is interesting to note that in general this function may generate a variety of moments that differ in the time order of operators involved, for instance,

$$Q_M^N = (-1)^M i^N \lim_{x^\pm \to 0} \frac{\partial^M}{\partial (x^-)^M} \frac{\partial^{N-M}}{\partial (x^+)^{N-M}} P\left(x^+, x^-, \tau\right)$$
$$= \int_0^\tau dt_1 ... dt_N \left\langle \overleftarrow{T} \{A(t_1) ... A(t_M)\} \times \overrightarrow{T} \{A(t_{M+1}) ... A(t_N)\} \right\rangle \cdot \quad (22)$$

The moments of (the not necessarily positive) $P(0, q, \tau)$ are expressed through these moments and binomial coefficients,

$$Q^{(N)} \equiv \int dq\, q^N P(0, q, \tau) = 2^{-N} \sum_M \binom{N}{M} Q_M^N. \quad (23)$$

6 FCS of a system in the ground state

To acquire a better understanding of the general relations obtained we consider now an important special case. We will assume that the system considered is in its ground state $|g\rangle$, so that its initial density matrix is $\hat{R} = |g\rangle\langle g|$. In this case the FCS is easily calculated. We have assumed that the coupling between the system and the detector is switched on adiabatically. Then the time development operators in (13) during the time interval $t_1 < t < 0$ adiabatically transfer the system from $|g\rangle$ into the ground state $|g(x^\pm)\rangle$ of the new Hamiltonian $\hat{H}_{\rm sys} - x^\pm \hat{A}$. In the time interval $0 < t < \tau$ the time evolution of the resulting state then has the simple form

$$e^{-it(\hat{H}_{\rm sys} - x^\pm \hat{A})}\, |g(x^\pm)\rangle \;=\; e^{-itE(x^\pm)}\, |g(x^\pm)\rangle \cdot \quad (24)$$

Here, $E(x^\pm)$ are the energies corresponding to $|g(x^\pm)\rangle$. This gives the main contribution to the FCS if the measurement time is large and the phase acquired during the switching of the interaction can be neglected in comparison with this contribution,

$$P\left(x^+, x^-, \tau\right) = e^{-i\tau\left[E\left(x^+\right) - E\left(x^-\right)\right]}. \quad (25)$$

We now assume the function $E(x)$ to be analytic and expand it in its Taylor series. We also re-scale q as above, $\bar{A} = q/\tau$. So, we have for the FCS

$$P\left(x, \bar{A}, \tau\right) = \int dz\; e^{-iz\bar{A}\tau} e^{-i\tau\left[E'(x)z + E'''(x)z^3/24 + ...\right]}. \quad (26)$$

First we observe that $P(x, \bar{A}, \tau)$ is a real function in this case, since the exponent in (26) is anti-symmetric in z. A first requirement for being able to interpret it as a probability distribution is therefore fulfilled. However, the same asymmetry assures that all *even* cumulative moments of $\bar{A}$ are identically zero, whereas the odd ones need not be. On the one hand, since the second moment corresponds to the noise and the ground state cannot provide any, this makes sense. On the other hand, this would be impossible if $P(0, \bar{A}, \tau)$ were a positive probability distribution unless it had no dispersion at all.

Belzig and Nazarov [8] encountered this situation analyzing the FCS of a super-conducting junction. In a certain limit the junction becomes a Josephson junction in its ground state. In this limit the interpretation of the FCS as a probability distribution does not work any longer. Fortunately enough, the relation (16) allows us to interpret the results obtained.

In the limit $\tau \to \infty$ of equation (26) terms involving higher derivatives of $E(x)$ are negligible and we have

$$\lim_{\tau \to \infty} P\left(x, \bar{A}, \tau\right) = \delta\left[\bar{A} + E'(x)\right]. \quad (27)$$

According to the Hellman-Feynman theorem $E'(x) = -\langle g(x)|\hat{A}|g(x)\rangle$. As one would expect, in this limit the measurement gives the expectation value of the operator $\hat{A}$ in a ground state of the system that is somewhat altered by its interaction with the detector at position x. Therefore the resulting dispersion of A will be determined by the *initial* quantum mechanical spread of the detector wave function. The error of the measurement stems from the interaction with the detector rather than from the intrinsic noise of the measured system.

7 FCS of electric current in a normal conductor

A complementary example is a normal conductor biased at finite voltage. This is a stationary *non-equilibrium* system far from being in its ground state. Here we do not intend to go to the microscopic details of the derivation. Our immediate aim is to make contact with the approaches of references [4,5]. We keep the original notations of the references wherever it is possible.

The starting points of the approaches differ much. Levitov and Lesovik propose a detector model where the z-component of a spin-1/2 creates a local vector potential felt by the electrons. This corresponds to a total Hamiltonian of the form

$$\hat{H} = \hat{H}_{\rm sys} - \frac{\lambda}{2e}\hat{\sigma}_z \hat{I}$$

which is studied at different coupling constants λ. Reference [5] starts with an extension of the Keldysh technique to only formally defined systems where the evolution of the wave function in different time directions is governed by two different Hamiltonians

$$\hat{H}^\pm = \hat{H}_{\rm sys} \pm \chi \hat{I} \quad (28)$$

and shows that the so defined Green functions can be used to generate moments of $\hat{I}$. This shall be compared with our detection model.

Despite different starting points, all three approaches quickly concentrate on the calculation of the quantity

$$\left\langle \exp\left[i\left(\hat{H}_{\rm sys} - x^+\hat{I}\right)\tau\right] \exp\left[-i\left(\hat{H}_{\rm sys} - x^-\hat{I}\right)\tau\right]\right\rangle \cdot \quad (29)$$

This quantity is denoted by $\chi(\lambda)$ in [4] and by $\exp[-S(\chi)]$ in [5]. It corresponds to our definition of the FCS equation (13) and we see now that the final result does not depend on the starting point.

As a concrete example we consider the FCS of the current in a phase-coherent conductor which is characterized by a set of transmission coefficients T_n (Eq. (37) of [3]). In general, the answer is expressed in terms of energy-dependent electron filling factors $n_{R(L)}$ on the right (left) side of the conductor,

$$\ln P\left(x^+,x^-,\tau\right) = \frac{\tau}{2\pi}\sum_n \int_{-\infty}^{+\infty} dE\,\ln\Big[1+T_n\left(e^{ie\left(x^- - x^+\right)}-1\right)n_R\left(1-n_L\right) +T_n\left(e^{ie\left(x^+ - x^-\right)}-1\right)n_L\left(1-n_R\right)\Big]. \quad (30)$$

This expression depends on $x^+ - x^-$ only. This is a direct consequence of gauge invariance. Indeed, in each of the Hamiltonians the coupling term is the coupling to a vector potential localized in a certain cross-section of the conductor. The gauge transform that shifts the phase of the wave functions by $ex^\pm$ on the right side of the conductor, eliminates this coupling term. This transform was explicitly implemented in [5]. Since there are *two* Hamiltonians in the expression, the coupling terms cannot be eliminated simultaneously provided that $x^+ \neq x^-$. However, the gauge transform with the phase shift $e(x^+ + x^-)/2$ makes the coupling terms depending on $x^+ - x^-$ only.

Since $P(x^+,x^-,\tau)$ depends on $x^+ - x^-$ only, the FCS $P(x,q,\tau)$ does not depend on x. As we have seen in Section 5, this enables one to interpret the FCS as a probability distribution.

Superconductivity breaks gauge invariance, thus making such an interpretation impossible.

8 FCS of a harmonic oscillator

Let us now illustrate the proposed measuring process with a simple example. We consider a detector that couples to the position of a harmonic oscillator in its ground state. The Hamiltonian of the system is then $\hat{H}_0 = \frac{\hat{Q}^2}{2M} + \frac{1}{2}M\omega^2\hat{X}^2$ and $\hat{A} = \hat{X}$ will be measured. The entire Hamiltonian in the measurement period reads then

$$\hat{H} = \frac{\hat{Q}^2}{2M} + \frac{1}{2}M\omega^2\hat{X}^2 - \hat{x}\,\hat{X}. \quad (31)$$

The perturbed ground state $|g(x)\rangle$ in this simple example is obtained by shifting the original ground state wave function by $x/M\omega^2$ in X-representation. Its energy is $E_g(x) = E_g(0) - \frac{1}{2M\omega^2}\,x^2$. We then find from (27), that

$$P(x,q,\tau) = \delta\left(q - x\tau/M\omega^2\right). \quad (32)$$

Following our first classical interpretation of $P(0,q,\tau)$ we would now conclude, that a harmonic oscillator in its ground state does not transmit any fluctuations of its position variable to the detector and that the detector's wave function is not altered by the oscillator. Calculating, however, the read-off of the detector with a Gaussian wave of uncertainty $\triangle q$ in the momentum as the initial state of the detector,

$$\rho^{in}(x,q) = \exp\left[-\frac{q^2}{2\left(\triangle q\right)^2} - 2\left(\triangle q\right)^2 x^2\right], \quad (33)$$

we find for the final momentum distribution

$$\Pi^f(q) = \exp\left[-\frac{q^2}{2\left(\triangle q\right)^2 + \tau^2/2M^2\omega^4\left(\triangle q\right)^2}\right]. \quad (34)$$

The uncertainty $\triangle q^f$ of the final detector momentum increases in time,

$$\left(\triangle q^f\right)^2 = \left(\triangle q\right)^2 + \frac{\tau^2}{4M^2\omega^4\left(\triangle q\right)^2}, \quad (35)$$

in contradiction to our first interpretation of equation (32). A $\triangle q^f$ that is growing with the detection time τ seems to imply that the detector does sense noise in the oscillator variable X. The true origin of this is, however, the interaction of the measured system with the detector. The detector position is spread over an interval $\triangle x \gtrsim 1/2\triangle q$. Since the oscillator is in its ground state the resulting disturbance drives it into ground states of new Hamiltonians $\hat{H}_0 + x\hat{X}$. For every detector influence x a different expectation value $E'(x) = \langle g(x)|\hat{X}|g(x)\rangle$ is measured. The read-off of the detector will then be a superposition of measurement outcomes corresponding to all those different influences. As a result, the uncertainty in the detector momentum grows with time, $(\triangle q^f)_{\triangle x} \approx \tau\triangle x\,\partial\langle g(x)|\hat{X}|g(x)\rangle/\partial x$. We conclude that the the quantum fluctuations of the detector set an upper bound on the accuracy of the measurement process. It vanishes if the FCS is x-independent and a classical interpretation of the process is possible.

9 Characterization of the FCS. First scheme

As we have already seen, the statistics $P(x,q,\tau)$ proposed above allows to predict the outcomes of measurements within our detection model and it resolves the inconsistencies that arose in earlier interpretations. It remains to be shown now, that it is real in the sense that it is experimentally observable.

For a first scheme of measuring the FCS we start from relation (16) between the initial and the final density matrix. Writing this equation in (x^+,x^-)-space, we find that

$$P(x^+,x^-,\tau) = \frac{\rho^f(x^+,x^-)}{\rho^{in}(x^+,x^-)} \quad (36)$$

or

$$P(x,q,\tau) = \int \frac{dz}{2\pi}\,e^{iqz}\,\frac{\rho^f(x+z/2,x-z/2)}{\rho^{in}(x+z/2,x-z/2)}. \quad (37)$$

We would already be done if we could measure all elements of the detector's final and initial density matrices. This is not possible in general, however. By successively measuring a certain observable we can measure the diagonal elements of the density matrix in a basis of eigenstates of that observable, but not the off-diagonal entries. We can therefore measure the functions $\Pi(q)$, but not $\hat{\rho}$ itself.

The key idea that we will pursue to solve this problem is to repeat the same measurement many times for shifted but otherwise identical initial detector density matrices. We suggest to repeat the measurement of the final momentum distribution $\Pi^f(q)$ for a number of initial density matrices that differ only in the expectation value x_0 of the position of the detecting particle and define the function

$$\Gamma^f(x_0,q,\tau) = \int dx\ dq'\ P(x,q-q',\tau)\ \rho^{in}(x-x_0,q'). \tag{38}$$

This way we expose the system during the measurement to different detector influences and one can hope that by doing so this influence can be identified and eliminated by a deconvolution procedure. Defining the Fourier transform of $\Gamma^f(x_0,q,\tau)$ with respect to both of its variables

$$\tilde{\Gamma}^f(q_0,z,\tau) \equiv \frac{1}{2\pi}\int dx_0\ dq\ e^{ix_0q_0-izq}\ \Gamma^f(x_0,q,\tau) \tag{39}$$

we find, that the FCS can indeed be reconstructed from this function by means of the relation

$$P(x,q,\tau) = \frac{1}{2\pi}\int dq_0\ dz\ e^{iqz-iq_0x}\ \frac{\tilde{\Gamma}^f(q_0,z,\tau)}{\tilde{\rho}^{in}(q_0,z)} \tag{40}$$

where

$$\tilde{\rho}(q_0,z) \equiv \int dx\ e^{iq_0x}\ \rho\left(x+\frac{z}{2},x-\frac{z}{2}\right)\cdot \tag{41}$$

To interpret the result of the measurement, we still have to know the full initial density matrix of the detector. This should be feasible, however. One might either prepare the detector initially in a specific, well-known state, or one might let the detector equilibrate with an environment. The initial density matrix is then stationary, $0 = [\hat{\rho}^{in},\hat{H}] \propto [\hat{\rho}^{in},\hat{q}^2]$, it is diagonal in a basis of momentum eigenstates and can be determined by a momentum measurement only. We conclude that the FCS is an observable.

To illustrate the procedure we apply it now to the example of a harmonic oscillator. The final momentum distribution with shifted initial detector states is

$$\Gamma^f(x_0,q,\tau) = \exp\left[-\frac{(q-x_0\tau/M\omega^2)^2}{(2(\triangle q)^2+\tau^2/2M^2\omega^4(\triangle q)^2)}\right]. \tag{42}$$

On transforming this into Fourier space it becomes

$$\tilde{\Gamma}^f(q_0,z,\tau) = \exp\left[-\frac{(\triangle q)^2z^2}{2}-\frac{\tau^2z^2}{8M^2\omega^4(\triangle q)^2}\right]\ \delta(q_0-\tau z/M\omega^2). \tag{43}$$

Employing now equation (40) with $\tilde{\rho}^{in}(q_0,z) = \exp\left[-(\triangle q)^2z^2/2-q_0^2/8(\triangle q)^2\right]$ we indeed recover the desired FCS equation (32).

10 Second scheme

If the system is in one state only, for example its ground state, or in a mixture of a limited number of discrete states, one can measure the FCS without knowledge of the initial detector state. We first assume that the system is in its ground state. Then we have the explicit expression (25) for the time evolution and we find, that

$$\Gamma^f(x_0,q,\tau) = \int dx\ dz\ dq' \times e^{-iz(q-q')-i\tau(E'(x)\ z+\frac{1}{24}E'''(x)\ z^3+\ldots)}\ \rho^{in}(x-x_0,q'), \tag{44}$$

$\Gamma^f(x_0,q,\tau)$ again being the final momentum distribution for shifted initial detector wave functions. In the limit of large τ we find with (27) that

$$\lim_{\tau\to\infty}\Gamma^f(x_0,q,\tau) \propto \int dx\ \rho^{in}(x-x_0,q+\tau E'(x)). \tag{45}$$

This formula suggests that one can measure the function $E'(x)$ arbitrarily exactly in the limit of a long measurement time τ by determining the peak of the final momentum distribution. The only assumption we have to make about the initial detector density matrix now is, that it is well centered around $x = 0$ and that it falls off sufficiently fast for momenta higher than some arbitrary $\triangle q$. We want $\hat{\rho}^{in}$ to be peaked in x-space such that $E'(x)$ is measured at the point x_0 only ($E(x)$ is assumed to be analytic). Of course, this means, that the width $\triangle q$ in momentum space of $\hat{\rho}^{in}$ and therefore also of $\hat{\rho}^f$ will be wide. For big τ, however, the peak position, that increases linearly in time, can still be detected with arbitrary precision.

Integrating $E'(x)$ we can then reconstruct the FCS for arbitrary detection times,

$$P(x,q,\tau) = \int \frac{dz}{2\pi}\ \exp\left\{-iqz-i\tau\int_{x-z/2}^{x+z/2}dx'\ E'(x')\right\}\cdot \tag{46}$$

When the system is in a mixture of N distinct states, the expression for $P(x,q,\tau)$ is a sum of terms of the form (25) with different functions $E_n(x)$. There appear in general N distinct peaks in the final momentum distribution allowing to record all N functions $E'_n(x)$. Again, one can reconstruct $P(x,q,\tau)$ for arbitrary τ.

11 Conclusions

We have studied the statistics of time averages of a quantum mechanical variable A. A simple model describing a detector without internal dynamics has been employed. The formalism that we have presented is, however, general enough to allow for the description of a generic detector. In our approach, the measurement outcome is expressed in terms of an object that we have called full counting statistics (FCS) of the variable A. It is an extension of another function proposed earlier in this context. This extension basically consists of accounting for the detector influence on the measured system. We find that the interplay of this influence with the quantum nature of the detector hampers in general a classical interpretation of the detector read-off. This way we have been able to remove inconsistencies ("negative probabilities") that arose in earlier interpretations [4]. Finally, we have shown, that this FCS is not only a theoretical construct that predicts results of measurements, but that it is an observable itself.

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Appendix A

Here we give a detailed derivation of the infinite mass limit of the kernel (10).

First, we define a Fourier transformed influence functional

$$\tilde{F}\left[k^+, k^-, \tau\right] = \int D\left[x^+\right] D\left[x^-\right] F\left[x^+, x^-, \tau\right] e^{i \int_{t_1}^{t_2} dt \left[x^+ k^+ - x^- k^-\right]} \tag{A.1}$$

and correspondingly

$$\tilde{K}\left(q^+, q^-, q_1^+, q_1^-, \tau\right) = \int dx^+ \, dx^- \, dx_1^+ \, dx_1^- \times e^{ix^+ q^+ - ix^- q^- - ix_1^+ q_1^+ + ix_1^- q_1^-} K\left(x^+, x^-, x_1^+, x_1^-, \tau\right). \tag{A.2}$$

The $k^\pm$ are functions on the interval $[t_1, t_2]$. Inserting the identity

$$\exp\left\{ i \int_{t_1}^{t_2} dt \, \frac{m}{2} \, \dot{x}^2 \right\} = \int D[q] \, \exp\left\{ i \int_{t_1}^{t_2} dt \, \left[-\frac{q^2}{2m} - q\dot{x} \right] \right\} \tag{A.3}$$

and using (10) we derive then

$$\tilde{K}(q^+, q^-, q_1^+, q_1^-, \tau) = \int \underset{q^+(t_2)=q^+, q^+(t_1)=q_1^+}{D[q^+]} \; \underset{q^-(t_2)=q^-, q^-(t_1)=q_1^-}{D[q^-]} \; \tilde{F}[\dot{q}^+, \dot{q}^-, \tau] \times \exp\left\{ i \int_{t_1}^{t_2} dt \, \frac{{q^+}^2}{2m} - \frac{{q^-}^2}{2m} \right\} \cdot \tag{A.4}$$

In the infinite mass limit, the kinetic term in this expression disappears. Now, we change integration variables from $D[q^\pm]$ to $D[\dot{q}^\pm]$ and call $k^\pm = \dot{q}^\pm$. Then,

$$\tilde{K}(q^+, q^-, q_1^+, q_1^-, \tau) = \int \underset{\int_{t_1}^{t_2} dt \, k^+ = q^+ - q_1^+}{D[k^+]} \; \underset{\int_{t_1}^{t_2} dt \, k^- = q^- - q_1^-}{D[k^-]} \; \tilde{F}[k^+, k^-, \tau]. \tag{A.5}$$

We can represent the functions $k^\pm$ by their Fourier series, $k^\pm(t) = \sum_{n=0}^{\infty} k_n^\pm \, \cos \, n\pi(t - t_1)/(t_2 - t_1)$. Changing the integration variables in (A.5) to the coefficients $k_n^\pm$ in this expansion, we notice, that only the integrals over the zeroth components $k_0^\pm$ are constrained by the boundary conditions. We can therefore do all the integrals over higher Fourier modes in (10) and obtain

$$\tilde{K}(q^+, q^-, q_1^+, q_1^-, \tau) = \int \underset{n\neq 0}{D[k_n^+]} \; \underset{n\neq 0}{D[k_n^-]} \; D[x_n^+] \, D[x_n^-] \, F[x^+, x^-, \tau] \times \exp\left\{ i \frac{t_2 - t_1}{2} \sum_{n=1}^{\infty} (k_n^+ x_n^+ - k_n^- x_n^-) + i\left[x_0^+(q^+ - q_1^+) - x_0^-(q^- - q_1^-)\right] \right\} \cdot \tag{A.6}$$

We have also expanded the functions $x^\pm$ in a Fourier series, $x^\pm(t) = \sum_{n=0}^{\infty} x_n^\pm \, \cos \, n\pi(t - t_1)/(t_2 - t_1)$.

We see, that the $k_n^\pm$-integrations result in δ-functions that constrain the $x_n^\pm$, $n \neq 0$, to zero and allow us to do the corresponding $x_n^\pm$ - integrals:

$$\tilde{K}(q^+, q^-, q_1^+, q_1^-, \tau) = \int dx_0^+ \, dx_0^- \, e^{i\left[x_0^+(q^+ - q_1^+) - x_0^-(q^- - q_1^-)\right]} \times \underset{\text{System}}{Tr} \, \overrightarrow{T} \, \exp\left\{ -i \int_{t_1}^{t_2} dt \left[\hat{H}_{\text{sys}} - \alpha_\tau x_0^+ \hat{A}\right] \right\} \times \hat{R} \, \overleftarrow{T} \, \exp\left\{ i \int_{t_1}^{t_2} dt \left[\hat{H}_{\text{sys}} - \alpha_\tau x_0^- \hat{A}\right] \right\} \cdot \tag{A.7}$$

When written in position space this relation is equivalent with equations (12) and (13). It establishes the locality of the kernel $K(x^+, x^-, x_1^+, x_1^-, \tau)$.

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